Impulse "Functions" (Dirac Delta)

When we discretize differential equations and solve the result, discrete impulse functions end up playing an important role.

Discrete impulse functions are the most basic and fundamental discrete functions.

The solution to a differential equation

$$\frac{d^n}{dt^n}y + p_1(t)\frac{d^{n-1}}{dt^{n-1}}y + \cdots + p_{n-1}(t)\frac{d}{dt}y + p_n(t)\,y = f(t)$$

where $f(t)$ = impulse function

is called an "impulse response" or "Green's function" — if you know all impulse responses for a DE then you can quickly write the solution for any function $f(t)$!

(This is called "discrete convolution" and is much easier than it sounds...)

Impulse "functions" are actually generalizations of functions (in the same way that $\sqrt{2}$ is a generalization of fractions).

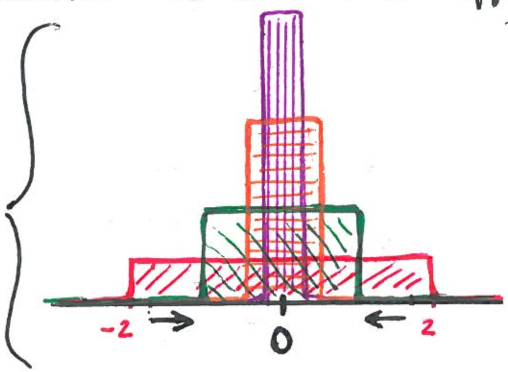
(Just like $\sqrt{2}$ is defined by the properties
- $(\sqrt{2})^2 = 2$
- $\sqrt{2} > 0$)

The impulse function $\delta(t)$ is defined by
- $\delta(t) = 0$ for $t \neq 0$
- $\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1$ for any $\varepsilon > 0$.

(Just like $\sqrt{2}$ is a limit of approximations
$\sqrt{2} = \text{limit}\ \{1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \ldots\}$)

The impulse function is a limit of approximations

$\delta(t) = $ limit



Discrete Impulse satisfies same properties

- $\delta_k = \delta(t_k) = 0$ for all $t_k \neq 0$
- $\underbrace{\int \underline{\delta}\, dt}_{} = 1$

← **Discrete Integral** $\sum_{k=0}^{n} \delta_k \cdot h$

⇝ First property gives δ_k for all but one k

⇝ Second property determines missing value

$$\sum_{k=1}^{n} \delta_k \cdot h = 1$$

$$(\cancel{\delta_0}^{0} + \cancel{\delta_1}^{0} + \cdots + \delta_i + \cdots + \cancel{\delta_n}^{0}) \cdot h =$$

$$\delta_i \cdot h = \qquad \text{(where } t_i = 0\text{)}$$

$$\Rightarrow \text{If } t_i = 0 \text{ then } \delta_i = \delta(t_i) = 1/h$$

EX: Discretize $\delta(t)$ on $[-1, 2]$ with $h = 1/2$

t	$\delta(t)$
$t_0 = -1$	$\delta_0 = 0$
$t_1 = -1/2$	$\delta_1 = 0$
$\boxed{t_2 = 0}$	$\boxed{\delta_2 = 2}$ ← Impulse point.
$t_3 = 1/2$	$\delta_3 = 0$
$t_4 = 1$	$\delta_4 = 0$
$t_5 = 3/2$	$\delta_5 = 0$
$t_6 = 2$	$\delta_6\ 0$

$$\underline{\delta} = 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note Always have $\underline{\delta} = 1/h \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← vector with
- one 1
- all others 0

EX: Discretize $\delta(t)$ on $[-1, 1]$ with $h = 1/3$

t	$\delta(t)$
$t_0 = -1$	$\delta_0 = 0$
$t_1 = -2/3$	$\delta_1 = 0$
$t_2 = -1/3$	$\delta_2 = 0$
$\boxed{t_3 = 0}$	$\boxed{\delta_3 = 3}$ ← Impulse Point
$t_4 = 1/3$	$\delta_4 = 0$
$t_5 = 2/3$	$\delta_5 = 0$
$t_6 = 1$	$\delta_6 = 0$

$$\underline{\delta} = 3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To get impulse at other points, we **shift** $\delta(t-c)$

⇝ move impulse point to $\underline{t_k = c}$

EX: Discretize $\delta(t-\underline{2})$ on $[0, 3]$ with $h = 1/2$

t	$\delta(t-2)$
$t_0 = 0$	$\delta_0 = 0$
$t_1 = 1/2$	$\delta_1 = 0$
$t_2 = 1$	$\delta_2 = 0$
$t_3 = 3/2$	$\delta_3 = 0$
$\boxed{t_4 = 2}$	$\boxed{\delta_4 = 2}$ ← Impulse Point
$t_5 = 5/2$	$\delta_5 = 0$
$t_6 = 3$	$\delta_6 = 0$

$$\underline{\delta} = 2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

It is very easy and natural to write discrete functions as sums of discrete impulses.

EX Discretize $f(t) = t^2$ on $[-1, 1]$ with $h = 1/2$ and write as a sum of discrete impulses.

t	$f = t^2$
$t_0 = -1$	$f_0 = 1$
$t_1 = -1/2$	$f_1 = 1/4$
$t_2 = 0$	$f_2 = 0$
$t_3 = 1/2$	$f_3 = 1/4$
$t_4 = 1$	$f_4 = 1$

$$\underline{\underline{f}} = \begin{bmatrix} 1 \\ 1/4 \\ 0 \\ 1/4 \\ 1 \end{bmatrix}$$

$$= 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1/4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 1/4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= 1/2 \cdot \underline{\underline{\delta(t-(-1))}} + 1/8 \, \underline{\underline{\delta(t-(-1/2))}} + 0 \cdot \underline{\underline{\delta(t)}}$$

$$+ 1/8 \cdot \underline{\underline{\delta(t-1/2)}} + 1/2 \cdot \underline{\underline{\delta(t-1)}}$$

(Note that with $h = 1/2$

$$\underline{\underline{\delta(t-c)}} = 2 \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$ ← vector of 0 with one 1)

→ Impulse functions are the "standard basis" for discrete functions!!!

(i.e. Writing a discrete function as a sum of impulses is like writing

$$\langle 1, -3, 2 \rangle = 1\,\vec{i} + (-3)\,\vec{j} + 2\,\vec{k}$$)

General Formula:

If $\underline{\underline{f}} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$ with step-size h then

$$\underline{\underline{f}} = (f_0 \cdot h)\, \underline{\underline{\delta(t-t_0)}} + (f_1 \cdot h)\, \underline{\underline{\delta(t-t_1)}} + \cdots$$

$$= \sum f_k \, \underline{\underline{\delta(t-t_k)}} \cdot h$$

Discrete convolution with impulse!

$$f = \int f(\tau)\, \delta(t-\tau)\, d\tau$$